Cambridge Assessment

A Level

Mathematics

Session: 1984 June

Type: Question paper

Code: 9200

SUMMER 1984

MATHEMATICS 1

9200/1, 9208/1

SYLLABUS A

PURE MATHEMATICS 1

ADVANCED LEVEL

(*Three hours*)

Answer **all** *the questions in Section A and any* **four** *questions from Section B.*

Mathematical tables, a list of formulae, and squared paper are provided.

Begin each answer to Section B on a fresh sheet of paper and arrange your answers in numerical order.

The intended marks for questions or parts of questions are given in brackets [ ].

Section A [52 *marks*]

Answer **all** *the questions in this section.*

1 Expand

$$\frac{2 + (1 + \frac{1}{2}x)^6}{2 + 3x}$$

in ascending powers of x up to and including the term in x^2. [6]

2 The (ordered) positive integers are bracketed thus:

$$(1);\ (2, 3);\ (4, 5, 6);\ (7, 8, 9, 10);\ \ldots,$$

there being r integers in the rth bracket. Find simplified expressions for

(i) the **last** integer in the nth bracket,

(ii) the **first** integer in the nth bracket.

[5]

3 Find the set of real values of x for which

$$|x + 3| > 2|x - 3|.$$ [6]

4 Solve for x the equation

$$\cos 2x + \sin x = 0,$$

giving in radians, in the form $k\pi$, all values of x lying between 0 and 2π. [6]

5 Given that $z = 2 - 3i$, express in the form $a + bi$ the complex numbers

(i) $(z + i)(z + 2)$,

(ii) $\dfrac{z}{1 - z^2}$.

[6]

6 Prove that each of the circles

$$x^2 + y^2 - 4x = 0$$

$$\text{and } x^2 + y^2 - 12x - 8y + 43 = 0$$

lies completely outside the other. [6]

7 A curve has parametric equations

$$x = t^2 + 1, \quad y = t^3.$$

Show that the point (5, −8) lies on the curve and obtain the equation of the tangent to the curve at this point. [5]

8 Show that

$$\int_0^{\frac{1}{3}\pi} \sin^3\theta \cos^2\theta \, d\theta = \frac{47}{480}.$$ [6]

9 Given that

$$\frac{dy}{dx} = (y + 2)\sec^2 x$$

and that $y = 1$ when $x = 0$, find an expression for y in terms of x. [6]

Section B

Answer any **four** *questions from this section. Each question in this section carries* 12 *marks. Begin each answer on a fresh sheet of paper and arrange your answers in numerical order.*

10 The expression

$$2x^4 + ax^3 + bx^2 - 4x - 4,$$

where a and b are constants, is denoted by $f(x)$. Given that $f(-\frac{1}{2}) = 0$ and $f(2) = 0$, find the values of a and b. [4]

With these values for a and b

(i) express $f(x)$ as the product of three algebraic factors, and hence show that the equation $f(x) = 0$ has only two real roots;

(ii) find the set of values of x for which $f(x) > 0$.

[8]

11 Express $\frac{3x}{(x-1)(x+2)}$ in partial fractions. [2]

Show that $\frac{dy}{dx}$ is negative at all points on the graph of

$$y = \frac{3x}{(x-1)(x+2)}.$$ [3]

Sketch this graph, showing the two asymptotes parallel to the y-axis and the asymptote perpendicular to the y-axis. [4]

By sketching on the same diagram a second graph (the equation of which should be stated), or otherwise, find the number of real roots of the equation

$$(x - 1)(x + 2)(x + 3) = 3x.$$ [3]

12 An equilateral triangle ABC has sides of length 6 units. The three altitudes of the triangle meet at N. Show that $AN = 2\sqrt{3}$ units. [3]

This triangle is the base of a pyramid whose apex V lies on the line through N perpendicular to the plane ABC. Given that $VN = 2$ units, prove that $V\hat{A}N = 30°$. [3]

The perpendicular from A to the edge VC meets CV produced at R. Prove that $AR = \frac{3}{2}\sqrt{7}$ units, and find the exact value of $\cos A\hat{R}B$. [6]

13 Prove that, for all values of θ,

$$\sin 3\theta - \cos 3\theta = (\sin\theta + \cos\theta)(2\sin 2\theta - 1).$$ [4]

Hence, or otherwise, find the values of θ, such that $0° \leqslant \theta \leqslant 180°$, for which

$$3(\sin 3\theta - \cos 3\theta) = 2(\sin\theta + \cos\theta).$$ [8]

[Where necessary give your answers correct to 0·1°.]

14 The point P on the parabola $y^2 = 4ax$ has coordinates $(ap^2, 2ap)$. The point S has coordinates $(a, 0)$. The line l has the equation $x + a = 0$. The line through P parallel to the x-axis meets l at the point M. Prove that $SP = PM$. [3]

The straight line PS meets the parabola again at Q $(aq^2, 2aq)$. Prove that $pq = -1$.

Prove also that the straight line MQ passes through the origin. [9]

15 The curves $ay = x^2$ and $y^2 = ax$, where $a > 0$, intersect at the origin O and the point A. Find the coordinates of A. [2]

Obtain, by integration, the area of the region R enclosed by the arcs of the two curves between O and A. [5]

Find the volume of the solid obtained when the region R is rotated through four right angles about the x-axis. [5]

16 Given that

$$y = (x^2 + 4)e^{-\frac{1}{2}x},$$

show that $\frac{dy}{dx} \leqslant 0$ for all values of x. [3]

Find the values of x for which

(i) $\frac{dy}{dx} = 0$,

(ii) $\frac{d^2y}{dx^2} = 0$.

[4]

Sketch the graph of

$$y = (x^2 + 4)e^{-\frac{1}{2}x},$$

showing, in particular, the form of the curve at the points where $\frac{dy}{dx} = 0$ and where $\frac{d^2y}{dx^2} = 0$. [5]

[It may be assumed that $y \to 0$ as $x \to \infty$.]

17 Show that the equation

$$(x + 1)^5 = (x + 2)^3 + 4$$

has a root between 0·8 and 1. [4]

Given that x_1 is a first approximation to this root, find, using the Newton-Raphson method of approximation, an expression for x_2, the next approximation, in terms of x_1. [4]

By taking x_1 equal to 1, use this method to evaluate successive approximations, ending when two successive approximations agree, after rounding, to three places of decimals. [4]

MATHEMATICS 2 **9200/2**

SYLLABUS A

ADVANCED LEVEL

(*Three hours*)

Answer any **seven** *questions.*

The intended marks for questions or parts of questions are given in brackets [ ].

Each question carries 14 *marks.*

Begin each answer on a fresh sheet of paper and arrange your answers in numerical order.

Mathematical tables, a list of formulae, and squared paper are provided.

Section A

Mechanics

1 (*a*) A particle is acted on by three forces with the following magnitudes and directions:

1N in the direction S,
2N in the direction N 60° W,
3N in the direction N 60° E.

Find the magnitude and direction of the resultant.

An additional force acts on the particle so that the particle is in equilibrium under the four forces. Find the magnitude and direction of this additional force. [8]

(*b*) Find k, given that

$$k\,\overrightarrow{BD} = 3\,\overrightarrow{AC} + 3\,\overrightarrow{BC} - 3\,\overrightarrow{AB} - \overrightarrow{BD} + 6\,\overrightarrow{CD},$$

where $ABCD$ is a plane quadrilateral. [6]

2 An aircraft, which can fly at 300 km h^{-1} in still air, has to fly directly from a point A to a point B, 600 km north-west of A. A steady wind is blowing from due south at a speed of 100 km h^{-1}. Using a graphical method, or any other method, find the course which the pilot should steer and the time taken for the journey. [7]

[Give your answers to an accuracy appropriate to your method.]

When the aircraft gets to B the pilot alters course and steers in a north-easterly direction. Find the distance and bearing of the plane, from B, two hours after leaving B. [7]

3 A uniform rectangular lamina has vertices A, B, C, D, where $AB = CD = 6a$, $AD = BC = 10a$. The point P lies on BC at a distance $6a$ from C and the point Q lies on AD at a distance $6a$ from D. The triangular portion CDP is folded through 180° about DP so that C lies on Q. Show that the centre of mass, G, of the folded lamina is at a perpendicular distance $\frac{22}{5}a$ from AB and find the perpendicular distance of G from AD. [8]

The folded lamina is suspended from P and hangs freely in equilibrium under gravity. Find the tangent of the angle which PQ makes with the vertical. [6]

4 A particle moves on the x-axis under the action of a force of variable magnitude directed along the x-axis. At time t the displacement of the particle from the origin O is x and the velocity of the particle in the positive direction of the x-axis is v.

(i) Given that the acceleration of the particle in the positive direction of the x-axis is $k^2(a - x)$, where k and a are positive constants, and that $v = ka\sqrt{3}$ when $x = 0$, show that $v^2 = k^2\,(a + x)\,(3a - x)$.

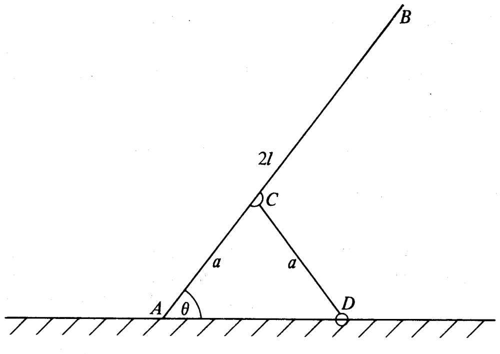
Find the maximum speed of the particle. [8]

(ii) Given that $v = Ve^{\alpha t}$, where V and α are positive constants, and that $x = 0$ when $t = 0$, find an expression for v in terms of x. [6]

5 From a point O, a particle P is projected under gravity with speed V, in a direction making an angle α with the horizontal. The particle strikes the horizontal plane through O at the point A. Find the time of flight and the distance OA. [4]

At the instant when P is at the highest point of its path a second particle Q is projected from O with speed U in a direction making an angle β with the horizontal. The particle Q strikes the plane at A at the same instant as P. Given that $\tan \alpha = \frac{4}{3}$, find U, in terms of V, and show that $\tan \beta = \frac{1}{3}$. [10]

6



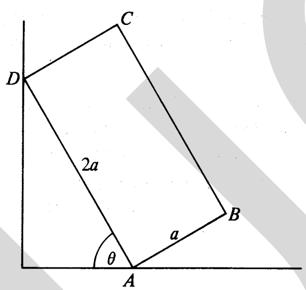
The diagram shows a uniform straight rod AB, of weight W and length $2l$, with the end A resting on a rough horizontal plane. A straight light rod CD, of length a, has one end freely hinged to the plane at D and the other end freely hinged to the rod AB at C, where $AC = a$. The rods rest in equilibrium in a vertical plane with AB inclined at an angle θ to the horizontal.

(i) Find the magnitude of the thrust in CD.

(ii) Find the horizontal and vertical components of the force on AB at A.

(iii) Show that $l \leqslant 2a$. [14]

7



The diagram shows a uniform rectangular lamina $ABCD$, of weight W, resting on horizontal ground at A and against a vertical wall at D. The lengths of AB and AD are a and $2a$ respectively. The coefficient of friction at A is $\frac{1}{2}$. The lamina is in equilibrium in a vertical plane perpendicular to the wall, with AD inclined at an angle θ to the horizontal.

(i) Given that the contact at D is smooth and that equilibrium is limiting, show that $\tan \theta = \frac{2}{3}$. [5]

(ii) Given that the coefficient of friction at D is $\frac{1}{3}$ and that equilibrium is limiting, find the value of $\tan \theta$. [9]

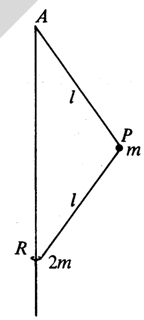
8 The engine of a car has maximum power P kW. The car has mass 1600 kg and a maximum speed of 30 m s^{-1} on a level road. The maximum speed up a hill of inclination α, where $\sin \alpha = \frac{1}{20}$, is 20 m s^{-1}. Given that when the speed is v m s^{-1} the resistance to motion has magnitude λv^2 N, where λ is a constant, show that the value of λ is $\frac{16}{19}$ and find the value of P, giving three significant figures in your answer. [Take the value of g to be 10 m s^{-2}.] [8]

Find the maximum acceleration of the car at an instant when it is travelling on a level road at a speed of 10 m s^{-1}. [6]

9 Three particles A, B and C, of masses m, $2m$ and $3m$ respectively, are at rest, in that order, in a straight line on a smooth horizontal table. The particles A and B are joined by a light inelastic string. Initially the string is slack and the particle B is projected away from A and towards C with speed V. The string becomes taut before B strikes C. Show that the loss of kinetic energy when the string becomes taut is $\frac{1}{3}mV^2$, and find the magnitude of the impulse in the string. [6]

The coefficient of restitution between B and C is $\frac{1}{2}$. Show that immediately after the collision between B and C the relative velocity of A and B has magnitude $\frac{3}{5}V$. [8]

10



The diagram shows a light inextensible string APR where $AP = PR = l$. The end A of the string is attached to a fixed point. A particle of mass m is attached to the string at P and a ring of mass $2m$ is attached at R. The ring is free to move on a fixed smooth vertical wire passing through A. The plane APR rotates with constant angular velocity ω about the vertical through A, and P moves in a fixed horizontal circle. Find the tension in AP, and show that $AR = 10g/\omega^2$.

Prove that $\omega^2 > 5g/l$. [14]

Section B

Statistics

11 A machine is designed to produce pieces of metal which are precisely 120 cm long. To test the accuracy of the machine, a random sample of 100 of these pieces of metal are measured, using a high precision optical measuring instrument, and the results are summarised below.

Length of metal in cm (x)	*Number of pieces*
$119{\cdot}996 < x \leqslant 119{\cdot}998$	10
$119{\cdot}998 < x \leqslant 120{\cdot}000$	50
$120{\cdot}000 < x \leqslant 120{\cdot}002$	30
$120{\cdot}002 < x \leqslant 120{\cdot}004$	10

(i) Display the results on a cumulative frequency graph. [4]

(ii) Estimate the median of this set of data, giving your answer correct to **four** decimal places. [3]

(iii) Determine the mean of this set of grouped data, giving your answer correct to **four** decimal places. [3]

(iv) Determine the variance of this set of grouped data, giving your answer correct to **three** significant figures. [4]

[Numerical results unsupported by evidence of method will receive little credit.]

12 On a particular railway the engines are of two classes, X and Y, there being 40 of class X and 20 of class Y. The engines are each distinguished by a different number. Out of the 60 engines on the railway, a train-spotter has, in the past, seen 25 different class X engines and 10 different class Y engines. Suppose that the next two engines that the train-spotter sees are equally likely to be any two of the 60 engines.

The events A and B are defined by

A: the train-spotter has seen both engines before,

B: the two engines belong to the same class.

Determine

(i) $P(A)$,

(ii) $P(B)$,

(iii) $P(A \cap B)$,

(iv) $P(A \cup B)$,

(v) $P(A \mid B)$.

[12]

State, with a reason, whether the events A and B are independent. [2]

13 On the island of Sans Serif, 50% of the (large) population are male and the remainder are female. Of the males, 20% have one pierced ear and the remainder have no pierced ears. Of the females, 60% have two pierced ears and the remainder have no pierced ears. Determine the expectation and variance of the number of pierced ears possessed by

(i) a male chosen at random,

(ii) a female chosen at random,

(iii) an islander chosen at random. [6]

An anthropologist, who visits the island, knows only that 50% of the population are female and 50% are male. The anthropologist has no knowledge of the distribution of pierced ears, and wishes to estimate the proportion p of ears that are pierced. The anthropologist can either examine the ears of a random sample of 20 islanders, or he can examine the ears of a sample comprising 10 randomly chosen males and 10 randomly chosen females. Show that both sampling methods give unbiased estimates of p and that the second method gives an estimate having a smaller variance. [8]

14 A man scatters pansy seeds and marigold seeds at random over a flower bed. The average density of the pansy seeds is 200 per square metre and that of the marigold seeds is 50 per square metre. The random variables X and Y denote the numbers of pansy seeds and marigold seeds, respectively, contained in a randomly selected portion of the flower bed of area $0{\cdot}01\ m^2$.

(i) Show that, correct to three decimal places, $P(X > 2) = 0{\cdot}323$.

(ii) Determine, correct to three decimal places, $P(X + Y = 0)$.

(iii) Show that $P(Y > 4) < 0{\cdot}001$.

(iv) Treating $P(Y > 4)$ as being negligible, determine, correct to three decimal places, $P(Y > X)$.

[14]

15 A target consists of a series of thin parallel straight lines, the distance between each pair of adjacent lines being $6r$. When a bullet strikes the target it makes a circular hole of radius r. The distance between the centre of a bullet hole and the nearest line may be assumed to be uniformly distributed between 0 and $3r$. A " hit " is obtained when the hole made by a bullet cuts a line.

(*a*) Five bullets have struck the target.

(i) Show that the probability that there have been exactly three hits is 40/243.

(ii) It is known that exactly two out of the first three bullets obtained hits. Determine the probability that exactly one of the remaining two bullets obtained a hit.

[9]

(*b*) Seventy-two bullets have struck the target. Using a suitable approximation, determine, correct to three decimal places, the probability that fewer than 20 of these 72 bullets have obtained hits. [5]

16 The continuous random variable X has a uniform distribution and takes values between 0 and 3 inclusive. The random variable Y is defined by $Y = X^2$. In any order,

(i) obtain the expectation and variance of Y, [6]

(ii) show that the distribution function of Y is given by

$$F(y) = \begin{cases} 0 & (y \leqslant 0), \\ \tfrac{1}{3}\sqrt{y} & (0 \leqslant y \leqslant 9), \\ 1 & (y \geqslant 9), \end{cases}$$

[5]

(iii) obtain the median of the distribution of Y. [3]

17 Two hikers, Adam and Ben, are independently journeying towards their common destination D. On a particular day Adam starts from A and Ben starts from B and both walk towards C. From C they will catch the first available bus to D. There are only 4 buses leaving C for D on that day, their departure times (unknown to Adam and Ben) being 10 30, 11 00, 11 30 and 12 00. The buses always leave on time. Adam leaves A at 08 00 and the time taken for his walk to C may be regarded as being an observation from a normal distribution with mean 3 hours and standard deviation 15 minutes. Show that, to three decimal places, the probability that Adam catches the 10 30 bus is 0·023 and the probability that he catches the 11 00 bus is 0·477. Find the probability that he catches (i) the 11 30 bus, (ii) the 12 00 bus. [5]

The time taken by the bus to reach D may be regarded as being an observation from a normal distribution with mean 32 minutes and standard deviation 2 minutes. Determine the probability that Adam arrives at D before 11 30. [4]

Ben leaves B at 09 00 and the time taken for his walk to C may be regarded as being an observation from a normal distribution with mean 1 hour 45 minutes and standard deviation 10 minutes. Assuming the times taken for their walks are independent of one another, determine the probability that Adam and Ben catch the same bus from C to D. [5]

[Give all your answers to three decimal places.]

18 An advertising consultant states that 30% of housewives prefer butter to margarine. In order to test this assertion, a random sample of 120 housewives were interviewed and 27 were found to prefer butter to margarine. Test whether these results provide significant evidence, at the 5% level, that the consultant was incorrect. State clearly your null and alternative hypotheses. [7]

An advertising campaign is launched to promote the consumption of butter. Following this campaign a second sample of 120 housewives were interviewed and n were found to prefer butter to margarine. Determine the smallest value of n which would provide significant evidence, at the 1% level, that more than 30% of housewives now prefer butter. [7]

19 A golfer practises her driving every morning of her holiday. The lengths (in metres) of a random sample of 40 of her drives at the start of her holiday are summarised by $\sum x = 7296$, $\sum(x-\bar{x})^2 = 28\,431$. Obtain an unbiased estimate of the variance of the length of her drives and obtain a symmetric 97% confidence interval for the mean length. [5]

The golfer hopes that her continued practice will increase the mean length of her drives. The lengths (in metres) of a random sample of 60 of her drives at the end of her holiday are summarised by $\sum y = 11\,631$, $\sum(y-\bar{y})^2 = 32\,819$. Assuming that the variance of the length of her drives has not been affected by her practice, obtain the unbiased two-sample estimate of this variance. Using this estimate of variance, and treating both samples as large samples, test whether there is significant evidence, at the 2% level, that the golfer's hope is justified. State your null and alternative hypotheses clearly. [9]

20 (*a*) The heights and weights of a random sample of 6 male A-level students are given in the table below:

Height in m	1·78	1·81	1·81	1·82	1·82	1·85
Weight in kg	64	58	70	66	75	75

(i) Plot these data on a scatter diagram.

(ii) Give the co-ordinates of the point of intersection of the regression line of height on weight with the regression line of weight on height.

(iii) Draw, by eye, the regression line of height on weight. Use your line to provide an estimate of the average height of male A-level students whose weight is 72 kg. [7]

(*b*) The heights and weights of a random sample of 100 male A-level students are summarised in the table below:

		Weight in kg (x)		
		$55 < x \leqslant 65$	$65 < x \leqslant 75$	$75 < x \leqslant 85$
Height	$1{\cdot}4 < y \leqslant 1{\cdot}6$	7	4	0
in m	$1{\cdot}6 < y \leqslant 1{\cdot}8$	17	35	12
(y)	$1{\cdot}8 < y \leqslant 2{\cdot}0$	1	11	13

(i) Calculate an estimate of the average height of male A-level students whose weight is 60 kg. Calculate the corresponding figures for students whose weight is 70 kg and for students whose weight is 80 kg.

(ii) Using the results of these calculations draw on a diagram the regression line of height upon weight [7]