# A Level

## Mathematics

Session: 1967 June
Type: Question paper
Code: 417

417/1
447/1

## MATHEMATICS 1

## PURE MATHEMATICS 1

ADVANCED LEVEL

(*Two hours and a half*)

*Answers to not more than* **eight** *questions are to be given up.*

*A pass mark can be obtained by good answers to about* **four** *questions or their equivalent.*

*Write on one side of the paper only, begin each answer on a fresh sheet of paper and arrange your answers in numerical order.*

*Mathematical tables, a list of formulae, and squared paper are provided.*

**1** (i) Show that if $\log a + \log c = 2\log b$ then $a$, $b$, $c$ are in geometric progression.

Show that if $\log x + \log z = 3\log y$ then $x$, $y^2$, $yz$ are in geometric progression.

(ii) Show that $x-y-z$ is a factor of the expression

$$x^3+y^3+z^3-yz(y+z)-zx(z+x)-xy(x+y)+2xyz.$$

Without further working write down two other factors of this expression.

**2** (i) Find the least positive integral value of $n$ for which

$$(\tfrac{4}{5})^n < \tfrac{1}{25}.$$

(ii) (*a*) Express with a rational denominator the fraction

$$\frac{\sqrt{c}+\sqrt{d}}{\sqrt{c}-\sqrt{d}}.$$

(*b*) Show that if $x$ is positive the value of $\sqrt{\{x(x+1)\}}$ lies between $x$ and $x+\frac{1}{2}$.

If $4x = p$, where $p$ is a positive integer, find, in terms of $p$, two consecutive integers between which the value of

$$\frac{\sqrt{(x+1)}+\sqrt{x}}{\sqrt{(x+1)}-\sqrt{x}}$$

lies.

**3** Show that as $x$ varies the maximum value of the function $(a-x)(x-b)$ is $\frac{1}{4}(a-b)^2$.

Illustrate this result by a sketch-graph of the function when $0 < b < a$.

If $0 < b < a$ find for the equation $(a-x)(x-b) = k$

(i) the value of $k$ (in terms of $a$ and $b$) for which the equation has equal roots,

(ii) the range of values of $k$ for which the equation has roots whose values lie between $b$ and $a$,

(iii) the range of values of $k$ for which both roots of the equation are positive.



**4** The series of natural numbers is grouped as follows:

$$(1),\quad (2, 3, 4),\quad (5, 6, 7, 8, 9),\quad \ldots$$

[i.e. each bracket contains two integers more than the preceding bracket].

(i) Find the total number of integers in the first $(n-1)$ brackets.

(ii) Show that the first number in the $n$th bracket is $n^2-2n+2$.

(iii) Show that the sum of the numbers in the $n$th bracket is $n^3+(n-1)^3$.

(iv) If the first number in the $n$th bracket is denoted by $a$ and the first number in the $(n+1)$th bracket is denoted by $b$ show that the sum of the numbers in the $n$th bracket is exactly divisible by $(b-a)$ and that the quotient is an odd number.

**5** Expand the functions $(1+x)^p$ and $\dfrac{1+ax}{1+bx}$ in ascending powers of $x$ as far as the terms in $x^3$.

If these expansions are identical as far as the terms in $x^2$, express $a$ and $b$ in terms of $p$.

By taking $p = \frac{2}{3}$, $x = \frac{1}{8}$ show that an approximation to the cube root of 81 is $\frac{212}{49}$.

Find, in terms of $p$ and $x$, the difference between the terms in $x^3$ of the two expansions, and evaluate this difference, correct to one significant figure, when $p = \frac{2}{3}$, $x = \frac{1}{8}$.

**6** (i) Use the table of cosines to find the three possible values of $\cos\theta$ when $\cos 3\theta = 0{\cdot}9$, and deduce the least possible value of $\cos 2\theta$, for values of $\theta$ which satisfy this equation.

(ii) Find all the angles between 0° and 360° which satisfy the equation

$$20\cos 3\theta = 9\sin 2\theta.$$

**7** In a circle, centre $O$, two radii $OP$, $OQ$ contain an angle $\theta$ radians ($\theta < \pi$). If the area of the sector $OPQ$ is $A$ and the length of the chord $PQ$ is $2c$ show that $1-\cos\theta = c^2\theta/A$.

Draw on the same diagram on squared paper the graphs of the functions $y = 1-\cos\theta$ and $y = c^2\theta/A$ in the particular case when $A = \pi$ sq. in., $c = 1\cdot3$ in., for values of $\theta$ between 0 and $\pi$. (Take $\frac{1}{2}$ in. to represent $\frac{1}{10}\pi$ on the $\theta$ axis and 2 in. to represent one unit on the $y$-axis.) Read off from your graphs the value of $\theta$ between 0 and $\pi$ which satisfies the equation $1-\cos\theta = c^2\theta/A$ in this particular case. Use your value of $\theta$ to find the radius of the circle in this case.

**8** Prove that $\cot\theta - \cot 2\theta = \operatorname{cosec} 2\theta$.

Replace $\theta$ by $2\theta$ in this identity and deduce that if $0° < \theta < 45°$ then $\cot\theta - \cot 4\theta > 2$.

The acute-angled triangle $ABC$ is such that $B = 4C$. By using the above result, or otherwise, prove that

$$b^2 - c^2 > 4\Delta,$$

where $\Delta$ is the area of the triangle.

**9** In the quadrilateral $ABCD$, $AB = 13$ in., $BC = 20$ in., $CD = 48$ in., $\angle BCD$ is 90° and $\angle BAC = \angle DBC$. Without using tables

(i) prove that $\cos BAC = \frac{5}{13}$,

(ii) prove that $\cos ACB = \frac{4}{5}$,

(iii) find the area of the quadrilateral by adding the areas of the triangles $ABC$ and $ACD$.

**10** From a point $P$ in a horizontal plane a man observes the summit $S$ of a mountain to bear due north at an elevation $\theta$. When the man has walked a distance $2a$ on a bearing $\alpha$ east of north to a point $Q$ in the horizontal plane he observes that the elevation of $S$ from $Q$ is again $\theta$. If $h$ is the height of $S$ above the horizontal plane containing $P$ and $Q$ show that $h = a\tan\theta\sec\alpha$.



When the man has walked a further distance $a$ in the same plane and in the same direction to a point $R$ he observes that the elevation of $S$ from $R$ is $\phi$. Show that

$$\cot^2\phi = (3\cos^2\alpha + 1)\cot^2\theta,$$

and that the distance $RS$ is

$$a(\sec^2\alpha\sec^2\theta + 3)^{\frac{1}{2}}.$$

**417/2**
**447/2**

# MATHEMATICS 2
# PURE MATHEMATICS 2

## ADVANCED LEVEL

*(Two hours and a half)*

*Answers to not more than* **eight** *questions are to be given up. A pass mark can be obtained by good answers to about* **four** *questions or their equivalent.*

*Write on one side of the paper only, begin each answer on a fresh sheet of paper and arrange your answers in numerical order.*

*Mathematical tables, a list of formulae, and squared paper are provided.*

**1** Without use of tables or accurate drawing find the gradients of the two lines through the origin which make an angle of 45° with the line $y = 2x$.

A square is to be constructed with one vertex at the origin $O$, and with one diagonal lying along the line $y = 2x$. If one of its sides (produced if necessary) is to pass through the point (3, 5) find, for each of the two possible squares, the equation of the second diagonal.

**2** The base $AB$ of a triangle $ABC$ is fixed, and $K$ is a fixed point on $AB$. The vertex $C$ of the triangle moves so that the perpendicular distances of $K$ from $CA$ and $CB$ are always equal in length. Prove that, in general, the locus of $C$ is a circle through $K$.

[If you use analytical methods, you are advised to take $A$ and $B$ on the $x$-axis, and to make $K$ the origin of co-ordinates.]

What is the exceptional case?

**3** Prove that the chord joining the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$ has the equation

$$(p+q)y = 2x+2apq.$$

Deduce that the chord $PQ$ goes through the focus $(a, 0)$ if $pq = -1$.

If $PQ$ and $RS$ are two chords of the parabola, each passing through the focus, prove that $PR$ and $QS$ meet on the directrix, i.e. on the line whose equation is $x+a = 0$.

**4** Prove that the normal at the point $P(a\cos\theta, b\sin\theta)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

has the equation

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta} = a^2-b^2.$$

The normal at $P$ cuts the axis of $x$ at $G$, and $PG$ is produced to $Q$ so that $GQ = 2PG$. Express the coordinates of $Q$ in terms of $a$, $b$ and $\theta$ and deduce that, as $\theta$ varies, $Q$ lies on a fixed ellipse. Give the $(x, y)$ equation of this ellipse.

**5** (i) Differentiate with respect to $x$

(*a*) $\dfrac{\cos x-\sin x}{\cos x+\sin x}$ (simplify your answer), (*b*) $\log_{10}x$.

(ii) Find constants $a$ and $b$ so that in the expansion of the function $x+a\sin x+bx\cos x$ in ascending powers of $x$, the first non-zero term is the one in $x^5$, and determine the co-efficient of that term.

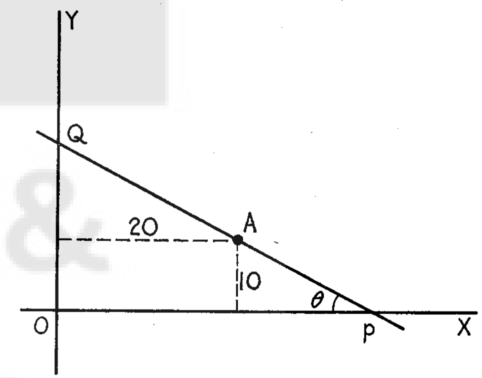


With the same values of $a$ and $b$ find the first non-zero term in the expansion of

$$1+(a+b)\cos x-bx\sin x$$

in ascending powers of $x$.

**6**




The point $A$ is distant 10 yards from a straight road $OX$, and 20 yards from a perpendicular straight road $OY$. A triangular piece of land $OPQ$ is formed, with $OP$, $OQ$ lying along $OX$ and $OY$ respectively, and $PQ$ passing through $A$. Angle $OPQ$ is denoted by $\theta$, and is acute. Find the perimeter of the triangle $OPQ$ in terms of $\theta$, and show that it has a stationary value when

$$2\cos\theta = 1+\sin\theta.$$

Verify that this equation is satisfied when $\cos\theta = \frac{4}{5}$, and find the corresponding value of the perimeter.

**7** If $y^2 = x^2(x-2)$, obtain an expression for $dy/dx$ in terms of $x$, and hence show that on the graph of $y$ against $x$ there are no turning points. Show that when $x = 2\frac{2}{3}$, $dy/dx = \pm\sqrt{6}$ and $d^2y/dx^2 = 0$.

Sketch the form of the graph of $y$ against $x$, paying special attention to the point $(2, 0)$ and to the points where $d^2y/dx^2 = 0$.

**8** (i) Evaluate

$$\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} 2\sin 3x \cos 2x \, dx,$$

giving your answer correct to two significant figures.

(ii) Using the substitution $t = \tan x$, or otherwise, find

$$\int \frac{dx}{4\cos^2 x - 9\sin^2 x}.$$

**9** The curves $y = 3\sin x$, $y = 4\cos x$ $(0 \leqslant x \leqslant \frac{1}{2}\pi)$ intersect at the point $A$, and meet the axis of $x$ at the origin $O$ and the point $B(\frac{1}{2}\pi, 0)$ respectively. Prove that the area enclosed by the arcs $OA$, $AB$ and the line $OB$ is 2 square units.

If $N$ is the foot of the perpendicular from $A$ to the axis of $x$, find by integration the volume obtained when the area enclosed by $AN$, $NB$ and the arc $AB$ is completely rotated about the $x$-axis, giving the answer correct to two significant figures.

**10** A body is moving in a straight line with a retardation proportional to the square of its velocity. With the usual notation express $dv/dt$ in terms of $v$.

Initially the body has a velocity of 2000 ft./sec. Find how far it has travelled after 3 sec., if its velocity is halved in that time.

[Give your answer in ft. correct to three significant figures.]

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## MATHEMATICS 3

## APPLIED MATHEMATICS 1

## STATISTICS 1

ADVANCED LEVEL

(*Three hours*)

*Candidates may choose their questions freely from the whole paper. Answers to not more than* **eight** *questions are to be*



*given up. A pass mark can be obtained by good answers to about* **four** *questions or their equivalent.*

*Write on one side of the paper only, begin each answer on a fresh sheet of paper and arrange your answers in numerical order.*

*Mathematical tables, a list of formulae, and squared paper are provided. In numerical work take g to be 32 ft. per sec.$^2$.*

### *Mechanics*

**1** A particle moves in a straight line, starting from a point $A$. Its motion is assumed to be with constant retardation. During the first, second and third seconds of its motion it covers distances of 70, 60 and 50 ft. respectively, measured in the same sense. Verify that these distances are consistent with the assumption that it is moving with constant retardation.

It comes instantaneously to rest at $B$. Find the distance $AB$.

At the same instant that the first particle leaves $A$, a second particle leaves $B$ with an initial speed of 75 ft. per sec. and travels with a constant acceleration along the line $BA$. If it meets the first particle at a point $C$, $1\frac{1}{2}$ sec. after leaving $B$, find the distance $BC$ and show that the acceleration of the second particle is 60 ft. per sec.$^2$.

**2** A particle is projected from a point $O$ on the horizontal ground with a velocity $V$ at an angle of elevation $\alpha$, and passes through a point $P$ which is at a distance of $x$ horizontally from $O$ and at a height $y$ above the ground. Prove that

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha.$$

A projectile is fired with a speed of $32\sqrt{10}$ ft. per sec. from the top of a vertical cliff 80 ft. high in a vertical plane at right angles to the line of the cliff and strikes the sea 160 ft. from the bottom of the cliff. Show that there are two possible directions of projection and that they are at right angles.

Show that the ratio of the times of flight for the two trajectories is $(2+\sqrt{5}):1$.

**3** If a body has two velocities, the magnitudes and directions of which are known, explain how to find the magnitude and direction of the resultant velocity.

If the magnitudes and directions of the velocities of two moving bodies $P$ and $Q$ are known, explain how to find the velocity of $Q$ relative to $P$.

A ship $P$, which can travel in still water at a maximum speed of 15 knots, is steering due north. It is moving at its maximum speed in a current flowing *from* the N.E. at 5 knots. Find graphically or otherwise the magnitude and direction of its resultant velocity.

A ship $Q$ is moving S.W. at 10 knots in an area where there is no current. Find graphically or otherwise the magnitude and direction of the velocity of $Q$ relative to $P$.

**4** If a particle describes a circle of radius $r$ with constant angular velocity $\omega$, prove that the magnitude of its acceleration is $r\omega^2$ and that the direction of its acceleration is towards the centre of the circle.

To one end of a light inextensible string of length $l$ is attached a particle of mass $m$. The particle is on a smooth horizontal table. The string passes through a smooth hole in the table and to its other end is attached a second particle of mass $m$.

The system is set in motion with the first particle describing a circle on the table with constant angular velocity $\omega_1$ and with the second particle moving in a horizontal circle as a conical pendulum with constant angular velocity $\omega_2$. Show that the lengths of the portions of the string on either side of the hole are in the ratio $\omega_2^2:\omega_1^2$.

Show also that the motion is only possible if

$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2} < \frac{l}{g}.$$



**5** A point $A$ is at a height $6a$ vertically above a point $B$. To a particle of mass $m$ are fastened two similar elastic strings, each of modulus $mg$ and of natural length $a$. The other ends of the strings are fixed, one at $A$ and one at $B$. Find the distance below $A$ of the point $P$ at which the particle will rest in equilibrium.

The particle is now lifted to the middle point of $AB$ and is there released. By considering the forces on the particle when it is at a distance $x$ below $P$, or otherwise, show that the particle performs simple harmonic motion. State the amplitude and the period of the simple harmonic motion.

**6** A boy fires a pellet of mass $\frac{1}{5}$ oz. at a speed of 84 ft. per sec. horizontally at a block of wood of mass 4 oz. which is lying at rest on a rough horizontal floor. The coefficient of friction between the block and the floor is $\frac{1}{2}$. The pellet adheres to the face of the block and the combined mass moves along the floor until it comes to rest. Find the initial speed of the combined mass and the distance it moves before coming to rest.

Calculate the impulse between the pellet and the block at the impact, stating the unit in which it is measured.

Show that the ratio of the energy lost at the impact to that lost during the subsequent motion is 20:1.

**7** $AOB$ is a diameter of the plane face of a solid uniform hemisphere of radius $a$, $O$ being the centre of its base. $OC$ is the radius at right angles to the plane face. A portion of the hemisphere in the form of a right circular cone is cut away, the circular base of the cone being in the plane face of the hemisphere, and the vertex $V$ of the cone being on $OC$. The radius of the base of the cone is $\frac{3}{4}a$ and its height is $\frac{2}{3}a$. Show that the distance from $O$ of the centre of gravity of the resulting solid $S$ is $\frac{11}{26}a$.

Find the weight which must be attached at $C$ so that, when the solid $S$, the weight of which is $W$, is freely suspended from $B$, the line $BV$ is vertical.

**8** A uniform rod $AB$ of weight $W$ and length $2a$ is smoothly hinged to a wall at $A$ and is held at an angle below the horizontal by means of an elastic string. The string is of modulus $kW$ and of natural length $a$ and connects the end $B$ of the rod to a point $C$ which is on the same horizontal level as $A$ and at a distance $2a$ from $A$. If the *string* makes an angle $\cos^{-1}\frac{4}{5}$ with the vertical, find the tension in the string in terms of $W$, and prove that $k = \frac{1}{8}$.

Find also the horizontal and vertical components of the reaction at $A$.

**9** Three equal weights are attached at the points $B$, $C$ and $D$ of a light string $ABCDE$ which hangs with $A$ and $E$ attached to fixed points at the same horizontal level. The system is in equilibrium with $A$ and $B$, $B$ and $C$, $C$ and $D$, $D$ and $E$ at the same horizontal distance $a$ apart, and with $C$ at a vertical distance $2a$ below $AE$. The system is symmetrical about the vertical through $C$. Find the tangents of the angles made with the horizontal by the portions of the string.

Show that the total length of the string is $(\sqrt{13}+\sqrt{5})a$.

**10** Two uniform rods $AB$ and $AC$, each of weight $W$ and length $4a$, are smoothly hinged together at $A$. At $B$ and $C$ are small light rings which run on a rough horizontal wire. The rods hang down below the wire, each making an acute angle $\theta$ with the vertical. A particle of weight $5W$ is fastened to the rod $AB$ at a point $D$, where $BD = a$. If the system is in equilibrium, find the force of friction and the normal reaction of the wire on each ring in terms of $W$ and $\theta$.

If the coefficients of friction at $B$ and $C$ are $\frac{1}{3}$ and 1 respectively, find the largest value of $\theta$ for which the system will stay in position without slipping.

### *Statistics*

**11** The following frequencies of the number ($r$) of disintegrations in a given time were obtained in 30 observations of a Wilson cloud chamber.



| Number of disintegrations ($r$) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Frequency ($f_r$) | 5 | 8 | 4 | 6 | 4 | 3 |

Plot these frequencies in a suitable diagram.

Calculate estimates of the mean and standard deviation of $r$. Obtain limits which you would expect with 95% confidence to enclose the population mean of $r$.

**12** A restaurateur has four times as many male customers as female; 40% of men and 70% of women take the set lunch, the remainder choosing from among the optional items of the menu. Of men choosing the set lunch, 10% drink wine, 50% beer and the remainder a soft drink, whilst of those choosing optional items the corresponding proportions are 60% and 30%. Among the women customers the corresponding proportions are 30% and 10% of those taking the set lunch and 40% and 20% of those choosing optional items.

(i) What proportion of customers take the set lunch?

(ii) What proportion of these drink wine?

(iii) What proportion of women customers drink beer?

(iv) What proportion of wine drinking customers are men?

**13** The proportion of defective articles produced in a certain manufacturing process has been found from long experience to be 0·1. In the first batch of 50 articles produced by a new process 3 were defective. Using the normal approximation to the binomial distribution, calculate the probability of so small a number of defective articles if the proportion of defective articles is unchanged, and explain how to use this probability in a significance test of the null hypothesis that the new process is no better than the old.

Calculate also the exact binomial probability without the use of the normal approximation.

**14** The variable $x$ is uniformly distributed in the range $a \leqslant x \leqslant b$. Specify precisely its frequency and cumulative frequency (distribution) functions.

What property of a distribution function corresponds to symmetry in a frequency function?

Show that if a variable $y$ is always positive its mean is equal to the area enclosed by the distribution function $F(y)$ and the lines $y = 0$, $F(y) = 0$, $F(y) = 1$. Suggest a convention for the sign of an area which could be used to allow this result to apply when $y$ is not always positive.

**15** The diameter of a mass produced rod is a normal variable with mean 0·85 cm. and standard deviation 0·03 cm. The internal diameter of the socket which holds the rod is a normal variable with mean and standard deviation respectively 0·94 cm. and 0·04 cm. Rods and sockets are paired randomly during assembly. What proportion of assemblies

(i) are rejected because the socket is too small for the rod?

(ii) are rejected because the rod is too loose in the socket, which is the case if the diameter of the socket exceeds that of the rod by more than 0·17 cm.?

If the diameter of the socket exceeds that of the rod by more than 0·12 cm. the assembly requires a spring washer. What proportion of accepted assemblies require a spring washer?

**16** The discrete variable $r$ takes the values $1, 2, 3, \ldots, N$ with probabilities $P_1, P_2, P_3, \ldots, P_N$. Interpret the statement

$$\sum_{r=1}^{N} P_r = 1.$$

Define the population mean ($\mu$) and population variance ($\sigma^2$) of $r$, and explain how to estimate these parameters from a sample of independent observations $r_1, r_2, \ldots, r_n$.

Prove that your estimate of $\sigma^2$ is unbiased. Is the square root of this estimate an unbiased estimate of $\sigma$?

[You may quote the result var $(\bar{r}) = \sigma^2/n$, where

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i.]$$



**17** The variable $x$ is normally distributed with mean $\mu$ and variance $\sigma^2$, both unknown. The values $\bar{x}_1$, $\bar{x}_2$ are respectively the means of independent samples of $n_1$ and $n_2$ observations. State the mean and variance of $X = a_1\bar{x}_1 + a_2\bar{x}_2$, where $a_1$ and $a_2$ are constants.

Show how to choose $a_1$ and $a_2$ so that the mean of $X$ is $\mu$ and the variance of $X$ is as small as possible.

State the mean and variance of $(\bar{x}_1 - \bar{x}_2)$ and hence or otherwise obtain an estimate of $\sigma^2$.

**18** The *moment generating function* (m.g.f.) of a variable $x$ is defined as $M(t) = E(e^{xt})$, where $E(\ )$ is the expectation operator. If $M(t)$ is known in the form of an expansion in ascending powers of $t$, show how the mean and variance of $x$ can be determined from it.

Find the m.g.f. of $x$ when the frequency function is $f(x) = \frac{1}{2}$, $1 \leqslant x \leqslant 3$. Hence find the variance of $x$ and verify your result from the definition of variance.

**19** The percentages of damaged cells in eight suspensions were determined by a differential staining technique and by phase-contrast microscopy with the following results

| Suspension | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Staining | 8·3 | 13·8 | 16·4 | 11·7 | 9·4 | 31·5 | 20·7 | 12·3 |
| Phase-contrast | 7·8 | 12·6 | 14·3 | 10·8 | 9·9 | 22·4 | 15·8 | 13·5 |

Plot the values obtained with the two techniques as ordinate and abscissa respectively and use the paired-sample $t$-test to compare the mean percentages. How far do you consider that this comparison is (*a*) valid, (*b*) helpful?

**20** An educational psychologist obtained scores by 9 university entrants in 3 tests ($A$, $B$ and $C$). The scores in tests $A$ and $B$ were as follows:

| Entrant | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $A$ score | 8 | 3 | 9 | 10 | 4 | 9 | 6 | 4 | 5 |
| $B$ score | 7 | 8 | 5 | 9 | 10 | 6 | 3 | 4 | 7. |

Calculate a coefficient of rank correlation between these two sets of scores.

The coefficients obtained between the $A$ and $C$ scores was 0·71 and that between the $B$ and $C$ scores was 0·62. What advice would you give the psychologist if he wished to use less than three tests?

*History of Mathematics*

**21** **Either** (*a*) Give a general account of the life and work of Sir Isaac Newton. Explain in detail the discoveries he made in **two** branches of mathematics.

**Or** (*b*) Explain how the people of the ancient (pre-Christian) world came to need mathematics. Describe **six** examples of the practical use of mathematics in these times.

**417/0**

## MATHEMATICS

SPECIAL PAPER

PAPER O

(*Three hours*)

*Candidates may attempt as many questions as they please, but marks will be assessed on the* **eight** *questions best answered.*

*Write on* **one** *side of the paper only, begin each answer on a fresh sheet of paper, and arrange your answers in numerical order.*

*Mathematical tables, a list of formulae, and squared paper are provided.*

**1** If $x$, $y$ and $z$ are any three real numbers, prove that

$$k(x^2+y^2+z^2)-2(yz+zx+xy)$$

can never be negative if $k \geqslant 2$.

Find conditions for the expression to vanish, distinguishing between the cases $k > 2$ and $k = 2$.



A closed rectangular box has total external surface area $A$, and the sum of the lengths of its twelve edges is $p$. The diagonal (i.e. the straight line joining the intersection of three faces to the intersection of the remaining three faces) is of length $d$.

Prove that

$$48d^2 > p^2 > 24A$$

unless the box is cubical.

**2** Find the range or ranges of values of $c$ such that the simultaneous equations

$$\cos\theta + \sin\phi = 1,$$
$$\sec\theta + \operatorname{cosec}\phi = c,$$

are satisfied by real values of $\theta$ and $\phi$.

Obtain the general solutions of these equations when $c = 6\frac{1}{4}$.

**3** The points $P_1$ and $P_2$ on the surface of the earth (assumed to be a sphere of radius $r$ and centre $O$) are at the same (Northern) latitude $\lambda$, and their respective longitudes are $L_1$ and $L_2$. Show that the length $C$ of the route from $P_1$ to $P_2$ via a circle centre $O$ (called a *great circle*) satisfies the equation

$$\cos\left(\frac{C}{r}\right) = \sin^2\lambda + \cos^2\lambda\cos(L_2-L_1).$$

If the separation in longitude is 90° show that either

$$C = 2r\sin^{-1}\left(\frac{\cos\lambda}{\sqrt{2}}\right),$$

or

$$C = 2\pi r - 2r\sin^{-1}\left(\frac{\cos\lambda}{\sqrt{2}}\right).$$

**4** A curve $C$ is defined parametrically by the equations $x = X(t)$, $y = Y(t)$. The point $Q$ with coordinates $(p, q)$ is a given point not lying on the curve. $P$ is the point on $C$ nearest to $Q$. Show that $t$, the parameter of $P$, satisfies the equation

$$\{p-X(t)\}\frac{dX(t)}{dt}+\{q-Y(t)\}\frac{dY(t)}{dt} = 0.$$

Deduce that $Q$ lies on the normal at $P$.

Find the co-ordinates of the point on the parabola

$$y^2 = 4ax$$

which is nearest to the point $(5a, 52a)$.

**5** If $y = \sin(m\sin^{-1}x)$, where $m$ is a constant, show that

(i) $y = 0$, when $x = 0$,

(ii) $dy/dx = m$, when $x = 0$,

(iii) $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y = 0$, for all values of $x$.

Taking $m = 3$, find constants $p$, $q$, $r$ and $s$ such that $y = p+qx+rx^2+sx^3$ satisfies conditions (i), (ii) and (iii).

Deduce an expression for $y = \sin(3\sin^{-1}x)$ and verify its correctness by another method.

**6** If $n$ is a non-zero integer and $m$ any constant not equal to $n$, show that

$$\int_0^{\frac{1}{2}\pi}\cos 2mx\cos 2nx\,dx = \frac{m(-1)^n}{2(m^2-n^2)}\sin m\pi.$$

Also find the value of the integral in the case when $m = n$.

Evaluate

$$\int_0^{\frac{1}{2}\pi}\cos^4 x\cos 4x\,dx.$$

**7** A derelict space-ship is losing heat by leakage of air at a rate proportional to its temperature $\theta$, and by radiation at a rate proportional to $\theta^4$. At time $t$, therefore, the temperature $\theta$ satisfies the differential equation

$$\frac{d\theta}{dt} = -k\theta-\frac{k}{c}\theta^4,$$

where $k$ and $c$ are constants.



If initially, at time $t = 0$, the temperature is $\theta_0$, prove that

$$\theta^3 = \frac{c\theta_0^3}{(c+\theta_0^3)\,e^{3kt}-\theta_0^3}.$$

Show also that the time required to cool to a temperature of $\frac{1}{2}\theta_0$ is

$$\frac{1}{3k}\log_e\left(\frac{8c+\theta_0^3}{c+\theta_0^3}\right).$$

**8** A wedge of mass $M$ with angles $\alpha$ and $\frac{1}{2}\pi$ (see Fig. 1) and base of length $b$ is at rest but free to move on a smooth horizontal table. A particle of mass $m$ is projected from a point of the table at the foot of the wedge with velocity $V$ in the vertical plane through the line of greatest slope, at an angle $\beta$ to the horizontal, where $\beta = \tan^{-1}(2\tan\alpha)$. Show that the particle strikes the wedge if

$$bg(1+4\tan^2\alpha) > 2V^2\tan\alpha,$$

and that it is moving horizontally at the time of impact.

If the particle adheres to the wedge, find their common velocity after impact.

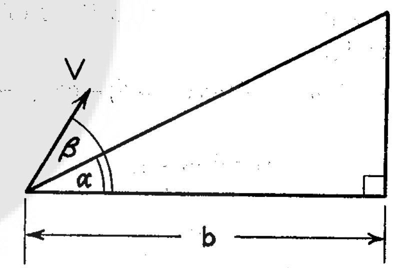


Fig. 1

**9** A bead $B$ of mass 1 gm. slides on a fixed smooth straight wire inclined at 30° to the horizontal. The point $A$ is situated at a height of 26 cm. vertically above a point $C$ on the wire. The bead is attracted to $A$ by a force directed along $BA$ and of magnitude $kBA$ dynes where $k = 100$ dynes/cm. Taking $g$, the acceleration due to gravity as 1000 cm./sec.², find the distance from $C$ of the position of equilibrium of the bead.

Show that if the bead is released from rest at $C$ at time $t = 0$, then at time $t$ its distance from $C$ is $16\sin^2 5t$.

Prove that the reaction of the wire on the bead is constant and find its value.

**10** A narrow tube is bent into the form of a circle, centre $O$, of radius $a$ which is fixed in a vertical plane, its highest point being $V$ (see Fig. 2). Two particles, $A$ (of mass $M$) and $B$ (of mass $m$), can slide without friction within the tube and are connected within the tube by a light inextensible string of length $\frac{1}{2}\pi a$. Initially the particles are at rest with the string taut, and $\angle VOA = \alpha$, where $M\tan\alpha > m$. Show that when $\angle VOA = \theta$ the tangential acceleration of the particles is

$$g\frac{(M\sin\theta - m\cos\theta)}{M+m}.$$

Prove that the tension in the string never exceeds

$$\frac{Mmg\sqrt{2}}{M+m}.$$

Using the principle of conservation of energy, or otherwise, show that when $\angle VOA = \theta$ the velocity $v$ of the particles satisfies the equation

$$(M+m)v^2 = 2Mga\sec\beta[\cos(\alpha-\beta) - \cos(\theta-\beta)],$$

where

$$\tan\beta = \frac{m}{M}.$$

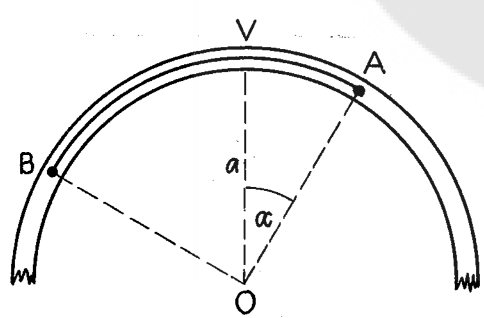


Fig. 2



**11** A thin straight rigid rod $AB$ of length $l$ is non-uniform, the weight per unit length increasing linearly from $w$ at $A$ to $2w$ at $B$. It rests in a vertical plane with the end $B$ in contact with a smooth vertical wall, and the end $A$ upon a rough horizontal floor. If the rod is inclined at $\theta$ to the horizontal, show that the coefficient of friction between the rod and the floor is not less than $\frac{5}{9}\cot\theta$.

**12** A cubical die with faces marked 1 to 6 is thrown $n$ times. Show that on the hypothesis that the die is unbiased the chance that the face marked 4 will be uppermost not more than once is $P$, where

$$P = \left(\frac{n+5}{5}\right)\left(\frac{5}{6}\right)^n.$$

If $n = 40$ and the face marked 4 comes uppermost exactly once, test whether the hypothesis that the die is unbiased is contradicted

(*a*) at the 1 % significance level,

(*b*) at the 0·1 % significance level.

**13** Half the population of the city of Ekron are Philistines and the other half are Canaanites. A Philistine never tells the truth but a Canaanite speaks truthfully with a probability $\frac{2}{5}$ and falsely with a probability $\frac{3}{5}$. What is the probability that a citizen encountered at random will give a correct answer to a question?

Tabulate (as fractions, decimals or percentages) the probabilities that 0, 1, 2 or 3 men out of a sample of 3 citizens taken at random from Ekron will affirm a proposition, (*a*) when it is true, (*b*) when it is false.

The proposition has a prior probability $\frac{3}{4}$ of being true. What is the (posterior) probability of it being true conditional on it being affirmed by only 1 out of the 3?

**14** Observation of a very large number of cars at a certain point on a motor-way establishes that the speeds are normally distributed. 90 % of cars have speeds less than 77·7 m.p.h., and only 5 % of cars have speeds less than 63·1 m.p.h. Determine the mean speed $\mu$ and the standard deviation $\sigma$.

**15** The following determinations of the speed of sound in air were made in independent repetitions of the same experimental procedure (the unit being $10^4$ cm./sec.).

3·35, 3·27, 3·30, 3·33, 3·32, 3·29.

Obtain an estimate based on all the determinations.

Also estimate $\sigma$, the standard deviation of a single determination.

Find an estimate of the standard deviation of your estimate of the speed of sound, and give (approximate) 95 % confidence limits for the speed of sound.

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## PURE MATHEMATICS 3

ADVANCED LEVEL

*(Three hours)*

*Answers to not more than* **nine** *questions are to be given up.*

*A pass mark can be obtained by good answers to about* **four** *questions or their equivalent.*

*Write on one side of the paper only, begin each answer on a fresh sheet of paper and arrange your answers in numerical order.*

*Mathematical tables, a list of formulae, and squared paper are provided.*

**1** (i) Prove that

$$\sum_{1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1),$$

and deduce the sum of the first $n$ terms of the series $1^2+3^2+5^2+7^2+\ldots$.

(ii) Sum to $n$ terms, and to infinity, the series

$$\frac{2}{1.3.5}+\frac{4}{3.5.7}+\frac{6}{5.7.9}+\ldots.$$



**2** Prove that, for real values of $x$, the function

$$\frac{(x+1)^2}{x^2+2}$$

lies between 0 and $\frac{3}{2}$, and state the limit of the function as $x$ tends to infinity.

Sketch its graph, indicating clearly its behaviour in relation to its asymptote.

**3** A man puts three £5 notes into one envelope and three £1 notes into a similar envelope. Each year, at Christmas, he chooses one envelope at random and gives his nephew a note from it. As soon as either envelope is emptied by his taking the last note from it the process ends.

(i) State the different totals which the nephew may have received when the process ends;

(ii) for each of these totals calculate the chance of its occurrence;

(iii) deduce that the nephew's expectation of gain is £12 7*s.* 6*d.*

**4** (i) Solve completely (in radians) the equation

$$\cot\theta - 2\cos 2\theta = 2.$$

(ii) Prove that

$$2\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{7}+2\tan^{-1}\tfrac{1}{8} = \tfrac{1}{4}\pi,$$

$\tan^{-1}$ here denoting the acute angle.

**5** (i) Define $\cosh x$ and $\sinh x$, and from these definitions prove that

$$\cosh 2x = 2\sinh^2 x + 1.$$

(ii) Prove that

$$(\cosh 2x+\sinh 2x-1)^n = 2^n \sinh^n x\,(\cosh nx+\sinh nx).$$

(iii) Solve the equation

$$2\cosh 2x - \sinh 2x = 2.$$